

# Maxima of the $Q$ -index: graphs with no $K_{s,t}$

Maria Agueiras A. de Freitas\*, Vladimir Nikiforov<sup>†</sup> and Laura Patuzzi<sup>‡</sup>

## Abstract

This note presents a new spectral version of the graph Zarankiewicz problem: How large can be the maximum eigenvalue of the signless Laplacian of a graph of order  $n$  that does not contain a specified complete bipartite subgraph. A conjecture is stated about general complete bipartite graphs, which is proved for infinitely many cases.

More precisely, it is shown that if  $G$  is a graph of order  $n$ , with no subgraph isomorphic to  $K_{2,s+1}$ , then the largest eigenvalue  $q(G)$  of the signless Laplacian of  $G$  satisfies

$$q(G) \leq \frac{n+2s}{2} + \frac{1}{2}\sqrt{(n-2s)^2 + 8s},$$

with equality holding if and only if  $G$  is a join of  $K_1$  and an  $s$ -regular graph of order  $n-1$ .

**Keywords:** *signless Laplacian; spectral radius; forbidden complete bipartite graphs; extremal problem.*

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## 1 Introduction.

*How many edges can a graph of order  $n$  have if it does not contain a given complete bipartite subgraph?* This variant of the famous Zarankiewicz problem [15] has turned out to be one of the most difficult problems in modern Discrete Mathematics, widely open despite long and intensive research. A comprehensive account of this vast theory can be found in the survey of Füredi and Simonovits [9].

For our presentation, let us write  $e(G)$  for the number of edges of a graph  $G$  and  $K_{s,t}$  for the complete bipartite graph with vertex classes of sizes  $s$  and  $t$ . Thus, the above problem can be stated as:

**Problem A.** *What is the maximum  $e(G)$  if  $G$  is a graph of order  $n$  containing no  $K_{s,t}$ ?*

Except for very few pairs of  $s$  and  $t$ , no general solution of Problem A is known. In a nutshell, the crucial difficulty is in the lack of constructions proving that the known upper bounds on  $e(G)$  are tight.

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\*Federal University of Rio de Janeiro, Brazil; email: [magueiras@im.ufrj.br](mailto:magueiras@im.ufrj.br)

<sup>†</sup>Department of Mathematical Sciences, University of Memphis, Memphis TN 38152, USA; email: [vnikifrv@memphis.edu](mailto:vnikifrv@memphis.edu)

<sup>‡</sup>Federal University of Rio de Janeiro, Brazil; email: [laura@im.ufrj.br](mailto:laura@im.ufrj.br)

Improving the long standing results of Kövari, Sós, and Turán [11] and of Znám [16], Füredi [7] gave a general bound, which was polished by Nikiforov in [14] to the final form stated in Theorem 2 below.

In fact, Nikiforov considered a spectral version of Problem A, which was also studied by Babai and Guiduli in [1]. Thus, let  $\lambda(G)$  denote the spectral radius of the adjacency matrix of a graph  $G$ . The following problem is the natural spectral analog of Problem A:

**Problem B.** *What is the maximum  $\lambda(G)$  of a graph  $G$  of order  $n$  containing no  $K_{s,t}$ ?*

An asymptotic upper bound was given by Babai and Guiduli in [1]; independently the following more precise statement was proved in [13] and [14]:

**Theorem 1** *Let  $s \geq t \geq 2$ , and let  $G$  be a  $K_{s,t}$ -free graph of order  $n$ , with spectral radius  $\lambda$ . If  $t = 2$ , then*

$$\lambda \leq 1/2 + \sqrt{(s-1)(n-1) + 1/4}. \quad (1)$$

*If  $t \geq 3$ , then*

$$\lambda \leq (s-t+1)^{1/t} n^{1-1/t} + (t-1) n^{1-2/t} + t-2. \quad (2)$$

Note that Theorem 1 is closely related to Problem A: indeed, the well-known inequality  $\lambda(G) \geq 2e(G)/n$  immediately implies Füredi's result [7], and even a slight improvement of it.

**Theorem 2** *Let  $s \geq t \geq 2$ , and let  $G$  be a  $K_{s,t}$ -free graph of order  $n$ , with  $e(G)$  edges. If  $t = 2$ , then*

$$e(G) \leq \frac{n}{2} \sqrt{(s-1)(n-1) + 1/4} + \frac{n}{4}. \quad (3)$$

*If  $t \geq 3$ , then*

$$e(G) \leq \frac{1}{2} (s-t+1)^{1/t} n^{2-1/t} + \frac{1}{2} (t-1) n^{2-2/t} + \frac{1}{2} (t-2) n.$$

It is worth pointing out that inequality (3) follows from a theorem of Hyltén-Cavallius [10] and is one of the few known tight results in the area, since Füredi constructed a matching family of graphs in [8].

**Theorem 3** *For any  $n$  there exist a  $K_{2,s+1}$ -free graph  $G$  of order  $n$  such that*

$$e(G) \geq \frac{1}{2} n \sqrt{sn} + O(n^{4/3}),$$

Another point to be made here is that equality holds in (1) and (3) if and only if  $G$  is a strongly regular graph with parameters

$$\left( n, 1/2 + \sqrt{(s-1)(n-1) + 1/4}, s, s \right).$$

Such strongly regular graphs are sometimes called *design graphs* (see, e.g., [2]) and appear in various problems.

The similarity between Theorems 1 and 2, together with the fact that most known constructions for Problem A are regular or almost regular graphs, suggests that Problems A and B might be essentially equivalent, and therefore equally hard. Let us note that such equivalence has been indeed proved for nonbipartite forbidden subgraphs. Thus, it is of interest whether other spectral versions of Problem A may be substantially different from Problem B, like the following one:

**Problem C.** *What is the maximum spectral radius of the signless Laplacian of a graph  $G$  of order  $n$  containing no  $K_{s,t}$ ?*

To make this question clear we need a brief introduction: let  $G$  be a graph with adjacency matrix  $A$  and let  $D$  be the diagonal matrix of the row-sums of  $A$ , i.e., the degrees of  $G$ . The matrix  $Q(G) = A + D$ , called the *signless Laplacian* or the  $Q$ -matrix of  $G$ , has been intensively studied, see, e.g., the survey of Cvetković [4] and its references. The maximal eigenvalue (equivalently, the spectral radius) of  $Q(G)$  is called the  $Q$ -index of  $G$  and is denoted by  $q(G)$ .

In general, in extremal problems with forbidden nonbipartite graphs,  $\lambda(G)$  and  $q(G)$  behave similarly (see a discussion of this fact in [6]), but they may be considerable differences between extremal problems about  $\lambda(G)$  and  $q(G)$  in case of forbidden bipartite graphs.

Regarding Problem C the authors believe it is not as difficult as Problems A and B, and it will be resolved completely in the next few years. Moreover, we venture the following conjecture:

**Conjecture 4** *Let  $s \geq t - 1 \geq 1$ , and let  $n$  be sufficiently large. If  $G$  is a  $K_{t,s+1}$ -free graph of order  $n$ , then*

$$q(G) \leq \frac{n}{2} + s + t - 2 + \frac{1}{2} \sqrt{(n - 2 + 2s)^2 - 8s(n - 2) + 4(t - 1)(n - t + 1)}.$$

*Equality holds if and only if  $G$  is a join of  $K_{t-1}$  and an  $s$ -regular graph of order  $n - t + 1$ .*

Presently we cannot prove or disprove this conjecture for all  $s$  and  $t$ . For  $t = 2$  and  $s = 1$  the graph  $K_{2,2}$  is just the cycle of length 4, and this case of Conjecture 4 was confirmed in [6]. In this paper we shall solve the case  $t = 2$  and any  $s \geq 1$ .

**Theorem 5** *Let  $s \geq 1$ , and let  $n \geq s^2 + 6s + 6$ . If  $G$  is a  $K_{2,s+1}$  graph of order  $n$ , containing no  $K_{2,s+1}$ , then*

$$q(G) \leq \frac{n + 2s}{2} + \frac{1}{2} \sqrt{(n - 2s)^2 + 8s}. \quad (4)$$

*Equality holds if and only if  $G$  is a join of  $K_1$  and an  $s$ -regular graph of order  $n - 1$ .*

We shall break Theorem 5 into two separate statements, with separate proofs. The purpose of this separation is twofold: first it helps with the presentation of the proof, and second it may be easier to analyze the proof and extend it for the general case of Conjecture 4.

We start with a result about a join of a vertex with a graph of bounded maximum degree. As proved in [5], if  $G$  is a join of  $K_1$  and an  $s$ -regular graph of order  $n - 1$ , then

$$q(G) = \frac{n + 2s}{2} + \frac{1}{2} \sqrt{(n - 2s)^2 + 8s}.$$

We shall give an easy improvement of this statement, making it an extremal result. Write  $G \vee H$  for the join of two graphs  $H$  and  $G$ .

**Proposition 6** *Let  $s \geq 1$ , let  $H$  be a graph of order  $n - 1$ , and let  $G := K_1 \vee H$ . If  $\Delta(H) \leq s$ , then*

$$q(G) \leq \frac{n + 2s}{2} + \frac{1}{2}\sqrt{(n - 2s)^2 + 8s} \quad (5)$$

*Equality holds if and only if  $H$  is  $s$ -regular.*

We postpone the proof of Proposition 6 to Section 2, after we introduce the necessary notation. Here we just note that simple as it is, Proposition 6 immediately takes care of the essential case of Theorem 5.

**Theorem 7** *Let  $s \geq 1$ , and let  $G$  be a graph of order  $n$ , with  $\Delta(G) = n - 1$ . If  $G$  is  $K_{2,s+1}$ -free, then*

$$q(G) \leq \frac{n + 2s}{2} + \frac{1}{2}\sqrt{(n - 2s)^2 + 8s}.$$

*Equality holds if and only if  $H$  is an  $s$ -regular graph.*

Theorem 7 obviously follows from Proposition 6, so we shall omit its proof.

The following theorem completes the proof of Theorem 5 and shows that if the premise  $\Delta(G) = n - 1$  is relaxed, we can strengthen the bound on  $q(G)$ .

**Theorem 8** *Let  $s \geq 1$ ,  $n \geq s^2 + 6s + 6$ , and  $G$  be a graph of order  $n$ , with  $\Delta(G) < n - 1$ . If  $G$  is  $K_{2,s+1}$ -free, then  $q(G) < n$ .*

Much of the rest of the paper is dedicated to the proof of Theorem 8, which is not too short.

## 2 Proofs of Proposition 6 and Theorem 8

First we shall introduce some notation; for graph notation undefined here we refer the reader to [3]. Thus, if  $G$  is a graph, and  $X$  and  $Y$  are disjoint sets of vertices of  $G$ , we write:

- $V(G)$  for the set of vertices of  $G$ ;
- $E(G)$  for the set of edges of  $G$ , and let  $e(G) := |E(G)|$ ;
- $\Delta(G)$  for the maximum degree of  $G$ ;
- $\Gamma(u)$  for the set of neighbors of a vertex  $u$ , and let  $d(u) := |\Gamma(u)|$ ;
- $G[X]$  for the graph induced by  $X$ , and let  $E(X) := E(G[X])$  and  $e(X) := |E(X)|$ ;
- $e(X, Y)$  for the number of edges joining vertices in  $X$  to vertices in  $Y$ .

Regarding the right side of (4), note that

$$\frac{n + 2s}{2} + \frac{1}{2}\sqrt{(n - 2s)^2 + 8s} > n,$$

and also

$$\begin{aligned} \frac{n + 2s}{2} + \frac{1}{2}\sqrt{(n - 2s)^2 + 8s} &= n + \frac{\sqrt{(n - 2s)^2 + 8s} - (n - 2s)}{2} \\ &= n + \frac{4s}{\sqrt{(n - 2s)^2 + 8s} + (n - 2s)} < n + \frac{2s}{n - 2s}. \end{aligned} \quad (6)$$

Part of our proof of Theorem 8 is based of the following inequality that can be traced back to Merris [12]:

For every graph  $G$ ,

$$q(G) \leq \max \left\{ d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v) \mid u \in V(G) \text{ and } d(u) > 0 \right\}. \quad (7)$$

**Proof of Proposition 6** Let  $w$  be the vertex of  $G$  corresponding to  $K_1$  in the representation  $G := K_1 \vee H$ . Set for short  $q := q(G)$  and let  $\mathbf{x} := (x_1, \dots, x_n)$  be a positive eigenvector to  $q$ . Choose a vertex  $u \in V(H)$  such that

$$x_u = \max_{a \in V(H)} x_a$$

From the eigenequations for the  $Q$ -matrix we have

$$qx_w = (n-1)x_w + \sum_{v \in V(H)} x_v \leq (n-1)x_w + (n-1)x_u \quad (8)$$

and

$$qx_u = d(u)x_u + \sum_{\{v,u\} \in E(G)} x_v \leq (s+1)x_u + x_w + sx_u.$$

Hence, we find that

$$(q - n + 1)x_w \leq (n-1)x_u, \quad (9)$$

$$(q - 2s - 1)x_u \leq x_w. \quad (10)$$

On the other hand, it is known that  $q \geq \Delta(G) + 1$ ; thus,  $q - n + 1 > 0$ . Therefore, we can multiply (9) and (10), obtaining

$$q^2 - (n+2s)q - 2s(n-1) \leq 0,$$

which implies (5).

If equality holds in (5), then equality holds in (8), and so  $x_v = x_u$  for any vertex  $v \in V(H)$ . Since for any  $v \in V(H)$  we have

$$qx_v = d(v)x_v + \sum_{\{p,v\} \in E(G)} x_p \leq (s+1)x_u + x_w + sx_u = qx_u$$

we see that  $d(v) = s+1$ , and so  $H$  is  $s$ -regular. ■

**Proof of Theorem 8** Suppose that  $G$  satisfies the hypothesis of the theorem, and for any nonisolated vertex  $u \in V(G)$ , let

$$F(u) := d(u) + \frac{1}{d(u)} \sum_{v \in \Gamma(u)} d(v).$$

Our first goal is to prove that if  $\Delta(G) \leq n-s-2$ , then  $F(u) < n$  for any nonisolated  $u \in V(G)$ , which, in view of (7), implies that  $q(G) < n$  as well.

If  $d(u) \leq s+1$ , then  $F(u) < n$  follows immediately by

$$d(u) + \frac{1}{d(u)} \sum_{s \in \Gamma(u)} d(s) \leq s+1 + \Delta(G) < n,$$

so from now on we shall suppose that  $s+2 \leq d(u) \leq n-s-2$ .

Fix a nonisolated vertex  $u \in V(G)$ , and let  $A := \Gamma(u)$  and  $B := V(G) \setminus (A \cup \{u\})$ . We see that

$$F(u) = d(u) + 1 + \frac{1}{d(u)} (2e(A) + e(A, B)).$$

Since  $G$  is  $K_{2,s+1}$ -free, we have  $\Delta(G[A]) \leq s$  and  $|\Gamma(v) \cap A| \leq s$  for any  $v \in B$ . Hence,

$$2e(A) \leq d(u)s$$

and

$$e(A, B) = \sum_{v \in B} |\Gamma(v) \cap A| \leq \sum_{v \in B} s = (n - d(u) - 1)s.$$

Adding the last two inequalities, we get the bound

$$F(u) \leq d(u) + 1 + \frac{(n-1)s}{d(u)},$$

so we want to prove that the right side is always less than  $n$ . Since the function

$$g(x) := x + 1 + \frac{(n-1)s}{x}$$

is convex for  $x > 0$ , and  $s+2 \leq d(u) \leq n-s-2$ , we see that

$$\begin{aligned} F(u) &\leq \max\{g(s+2), g(n-s-2)\} \\ &= \max\left\{s+3 + \frac{(n-1)s}{s+2}, n-s-1 + \frac{(n-1)s}{n-s-2}\right\} \end{aligned}$$

After some simple algebra, we find that if  $n > s^2 + 2s + 2$ , then

$$\max\left\{s+3 + \frac{(n-1)s}{s+2}, n-s-1 + \frac{(n-1)s}{n-s-2}\right\} < n,$$

completing the proof if  $\Delta(G) \leq n-s-2$ .

Thus, it remains to prove the theorem for  $\Delta(G) \geq n-s-1$ . In this case we shall use completely different approach. Assume for a contradiction that  $q(G) > n$ . Set for short  $q := q(G)$  and let  $\mathbf{x} := (x_1, \dots, x_n)$  be a nonnegative unit eigenvector to  $q$ .

Choose a vertex  $w$  with  $d(w) = \Delta(G)$  and let  $A = \Gamma(w)$  and  $B = V(G) \setminus (\{w\} \cup A)$ . Note that

$$|B| = n - 1 - d(w) = n - 1 - \Delta(G) \leq s.$$

Now, choose  $u \in A$  and  $v \in B$  such that

$$x_u = \max_{a \in A} x_a \quad \text{and} \quad x_v = \max_{a \in B} x_a.$$

Our first goal is to show that  $d(u) \leq 2s + 1$  and  $d(v) \leq 2s - 1$ . This follows easily by

$$d(u) = 1 + |\Gamma(u) \cap A| + |\Gamma(u) \cap B| \leq 1 + s + |B| \leq 2s + 1,$$

and likewise,

$$d(v) = |\Gamma(v) \cap B| + |\Gamma(v) \cap A| \leq |B| - 1 + s \leq 2s - 1.$$

Next, we shall show that  $x_v < 2/n$  and  $x_u < 2/n$ . To this end note that the eigenequations for  $Q(G)$  corresponding to  $u$  and  $v$  imply that

$$qx_u = d(u)x_u + x_w + \sum_{a \in \Gamma(u) \cap A} x_a + \sum_{a \in \Gamma(u) \cap B} x_a < (2s + 1)x_u + 1 + sx_u + sx_v,$$

and

$$qx_v = d(v)x_v + \sum_{a \in \Gamma(v) \cap A} x_a + \sum_{a \in \Gamma(v) \cap B} x_a \leq (2s - 1)x_v + sx_u + (s - 1)x_v.$$

Rearranging these inequalities, we get

$$(q - 3s - 1)x_u < 1 + sx_v \tag{11}$$

and

$$(q - 3s + 2)x_v \leq sx_u.$$

Now, excluding  $x_u$ , we find that

$$(q - 3s + 2)x_v < \frac{(1 + sx_v)s}{q - 3s - 1} \leq \frac{(1 + sx_v)s}{n - 3s - 1} < \frac{s}{n - 3s - 1} + sx_v,$$

and so,

$$x_v < \frac{s}{(n - 3s - 1)(n - 4s + 2)} \leq \frac{1}{(s + 1)(n - 4s + 2)} \leq \frac{1}{2(n - 4s + 2)} \leq \frac{2}{n}.$$

To bound  $x_u$  we substitute  $2/n$  for  $x_v$  in (11) and obtain

$$x_u < \frac{1 + sx_v}{q - 3s - 1} < \frac{1 + 2s/n}{n - 3s - 1} < \frac{1 + 1/3}{n - 3s - 1} \leq \frac{2}{n}.$$

Armed with the upper bounds on  $x_u$  and  $x_v$ , we shall prove that  $q < n$ . We shall use some relatively new techniques for this purpose. To begin with, since  $\mathbf{x}$  is a unit vector, we have

$$q = \langle Q\mathbf{x}, \mathbf{x} \rangle = \sum_{\{i,j\} \in E(G)} (x_i + x_j)^2.$$

Write  $G'$  for the graph  $G[A \cup \{w\}]$  and set  $n' = |V(G')|$ . Note that

$$\sum_{\{i,j\} \in E(G)} (x_i + x_j)^2 = \sum_{\{i,j\} \in E(G')} (x_i + x_j)^2 + \sum_{\{i,j\} \in E(B)} (x_i + x_j)^2 + \sum_{\{i,j\} \in E(A,B)} (x_i + x_j)^2 \quad (12)$$

Since  $\Delta(G') = n' - 1$ , Theorem 7, together with (6), implies that

$$\sum_{\{r,s\} \in E(G')} (x_r + x_s)^2 \leq \frac{n' + 2s + \sqrt{(n' - 2s)^2 + 8s}}{2} < n' + \frac{2s}{n' - 2s}.$$

On the other hand, using the inequalities  $x_v < 2/n$  and  $x_u < 2/n$ , we see that

$$\begin{aligned} \sum_{\{i,j\} \in E(B)} (x_i + x_j)^2 + \sum_{\{i,j\} \in E(A,B)} (x_i + x_j)^2 &\leq \sum_{\{i,j\} \in E(B)} (x_v + x_v)^2 + \sum_{\{i,j\} \in E(A,B)} (x_u + x_v)^2 \\ &\leq e(B) \left( \frac{2}{n} + \frac{2}{n} \right)^2 + e(A, B) \left( \frac{2}{n} + \frac{2}{n} \right)^2 \\ &< \left( \binom{s}{2} + s^2 \right) \frac{16}{n^2} < \frac{24s^2}{n^2}. \end{aligned}$$

Therefore, in view of (6) and (12), we obtain

$$q \leq n' + \frac{2s}{n' - 2s} + \frac{24s^2}{n^2}.$$

Since the function  $g(x) = x + 2s/(x - 2s)$  is convex whenever  $x > 2s$ , the inequalities

$$n - s \leq n' \leq n - 1$$

imply that

$$n' + \frac{2s}{n' - 2s} \leq \max \left\{ n - 1 + \frac{2s}{n - 1 - 2s}, n - s + \frac{2s}{n - 3s} \right\}.$$

In view of  $s \geq 1$  and  $n > 3s + 2$ , one easily finds that

$$n - s + \frac{2s}{n - 3s} \leq n - 1 + \frac{2s}{n - 1 - 2s}.$$

Therefore,

$$\begin{aligned} q &\leq n - 1 + \frac{2s}{n - 1 - 2s} + \frac{24s^2}{n^2} \leq n - 1 + \frac{2}{s + 4 + 5/s} + \frac{24}{(s + 6 + 6/s)^2} \\ &< n - 1 + \frac{2}{5} + \frac{24}{49} < n. \end{aligned}$$

The proof of Theorem 8 is completed. ■



### 3 Concluding remarks

In our proof of Theorems 5 and 8 we used techniques that have worked efficiently for solving a number of extremal problems about the  $Q$ -index; however, these methods seem inadequate for tackling Conjecture 4 in general. We need completely new general techniques, for which Conjecture 4 provides both motivation and a test field.

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